Kalkulus tizenhetedik feladatsor - megoldás

Parciális integrálás, racionális törtfüggvények integrálása

1. Számítsa ki az alábbi határozatlan integrálokat!

   a) $\int x\cos(2x)\,\mathrm{dx}$
   b) $\int (x^2+1)e^x\,\mathrm{dx}$
   c) $\int e^x\sin(1-x)\,\mathrm{dx}$
   d) $\int x^3\ln(2x)\,\mathrm{dx}$ (K87)
   e) $\int (3x-1)\sin(5x+3)\,\mathrm{dx}$ (K87)
   f) $\int \arctan(2x)\,\mathrm{dx}$ (K87)
   g) $\int \cosh(2x)\sin(5x)\,\mathrm{dx}$ (K88)
   h) $\int \ln(5x)\,\mathrm{dx}$
   i) $\int (2x+5)\ln(5x)\,\mathrm{dx}$
   j) $\int (5x+2)\sinh(4x)\,\mathrm{dx}$
   k) $\int x^2\cos(3x)\,\mathrm{dx}$
   l) $\int \arcsin(2x)\,\mathrm{dx}$
   m) $\int 4x\arctan(2x)\,\mathrm{dx}$

   Megoldás: A parciális integrálás képlete: $\int f'g = fg - \int fg'$

   h) $\int \ln(5x)\,\mathrm{dx} = \int 1\cdot\ln(5x)\,\mathrm{dx} = \int \overbrace{1}^{f'}\cdot\overbrace{\ln(5x)}^{g}\,\mathrm{dx} = x\ln(5x) - \int x\frac{1}{5x}5\,\mathrm{dx} = x\ln(5x) - \int 1\,\mathrm{dx} = x\ln(5x) - x + C$

   i) $\int (2x+5)\ln(5x)\,\mathrm{dx} = \int \overbrace{(2x+5)}^{f'}\cdot\overbrace{\ln(5x)}^{g}\,\mathrm{dx}$, *Ekkor* $f(x) = x^2+5x$
   $\int (2x+5)\ln(5x)\,\mathrm{dx} = \left(x^2+5x\right)\ln(5x) - \int \left(x^2+5x\right)\frac{1}{5x}5\,\mathrm{dx} = \left(x^2+5x\right)\ln(5x) - \int x+5\,\mathrm{dx} = \left(x^2+5x\right)\ln(5x) - \left(\frac{x^2}{2}+5x\right) + C$

   j) $\int (5x+2)\sinh(4x)\,\mathrm{dx} = \int \overbrace{(5x+2)}^{g}\cdot\overbrace{\sinh(4x)}^{f'}\,\mathrm{dx}$ *Ekkor* $f = \frac{\cosh(4x)}{4}$
   $\int (5x+2)\sinh(4x)\,\mathrm{dx} = (5x+2)\frac{\cosh(4x)}{4} - \int \frac{\cosh(4x)}{4}5\,\mathrm{dx} = (5x+2)\frac{\cosh(4x)}{4} - \frac{\sinh(4x)}{16}5 + C$

   k) $\int x^2\cos(3x)\,\mathrm{dx} = \int \overbrace{x^2}^{g}\cdot\overbrace{\cos(3x)}^{f'}\,\mathrm{dx}$ *Ekkor* $f = \frac{\sin(3x)}{3}$,
   $\int x^2\cos(3x)\,\mathrm{dx} = x^2\frac{\sin(3x)}{3} - \int 2x\frac{\sin(3x)}{3}\,\mathrm{dx}$

   $\int 2x\frac{\sin(3x)}{3}\,\mathrm{dx} = \frac{2}{3}\int \overbrace{x}^{g}\cdot\overbrace{\sin(3x)}^{f'}\,\mathrm{dx}$ *Ekkor* $f = -\frac{\cos(3x)}{3}$,
   $\int 2x\frac{\sin(3x)}{3}\,\mathrm{dx} = \frac{2}{3}\left(-\frac{x\cos(3x)}{3} - \int -\frac{\cos(3x)}{3}\,\mathrm{dx}\right) =$
   $= -\frac{2x\cos(3x)}{9} + \frac{2\sin(3x)}{27} + C$
   $\int x^2\cos(3x)\,\mathrm{dx} = x^2\frac{\sin(3x)}{3} + \frac{2x\cos(3x)}{9} - \frac{2\sin(3x)}{27} + C$

l) $\int \arcsin(2x)\,\mathrm{dx} = \int 1 \cdot \arcsin(2x)\,\mathrm{dx} = \int \overbrace{1}^{f'} \cdot \overbrace{\arcsin(2x)}^{g}\,\mathrm{dx} =$

$= x\arcsin(2x) - \int x \frac{1}{\sqrt{1-(2x)^2}} 2dx = x\arcsin(2x) - (-\frac{1}{4}) \int \underbrace{(-8x)}_{f'} \underbrace{\left(1-4x^2\right)^{-\frac{1}{2}}}_{f^\alpha}\,\mathrm{dx} =$

$= x\arcsin(2x) + \frac{1}{4}\frac{(1-4x^2)^{\frac{1}{2}}}{\frac{1}{2}} + C = x\arcsin(2x) + \frac{1}{2}(1-4x^2)^{\frac{1}{2}} + C$

m) $\int 4x\arctan(2x)\,\mathrm{dx} = \int \overbrace{4x}^{f'} \cdot \overbrace{\arctan(2x)}^{g}\,\mathrm{dx}$ *Ekkor* $f = 2x^2$

$\int 4x\arctan(2x)\,\mathrm{dx} = 2x^2\arctan(2x) - \int 2x^2 \frac{1}{1+(2x)^2} 2\,\mathrm{dx} = 2x^2\arctan(2x) -$

$\int \frac{4x^2+1-1}{1+4x^2}\,\mathrm{dx} == 2x^2\arctan(2x) - \int 1 - \frac{1}{1+4x^2}\,\mathrm{dx} =$

$= 2x^2\arctan(2x) - x + \frac{\arctan(2x)}{2} + C$

2. Számítsa ki az alábbi racionális törtfüggvények integrálját! *(Kónya 5.3 fejezet.)*

 a) $\int \frac{1}{x^2+2x-3}\,\mathrm{dx}$
 b) $\int \frac{x+1}{x^2+3x}\,\mathrm{dx}$ (Mo: Kónya 89.o.)
 c) $\int \frac{2x+1}{x^2-5x+6}\,\mathrm{dx}$ (Mo: Kónya 90.o.)
 d) $\int \frac{1}{x^3+2x^2}\,\mathrm{dx}$ (Mo: Kónya 90.o.)
 e) $\int \frac{x+1}{(x-1)^2(x-3)}\,\mathrm{dx}$ (Mo: Kónya 91.o.)
 f) $\int \frac{x^3}{x^4-16}\,\mathrm{dx}$ (Mo: Kónya 91.o.)
 g) $\int \frac{x^5-15x}{x^4-16}\,\mathrm{dx}$ (Mo: Kónya 91.o.)

 Megoldás: A polinomosztáshoz: $(x^5 - 15x) : (x^4 - 16)$

$$\begin{array}{l} (\quad x^5 - 15x) \div (x^4-16) = x + \dfrac{x}{x^4-16} \\ \underline{-x^5 + 16x} \\ \qquad\quad x \end{array}$$

 Másik példa: $(x^3 - 1) : (x - 1)$

$$\begin{array}{l} (\quad x^3 \qquad\quad - 1) \div (x-1) = x^2 + x + 1 \\ \underline{-x^3 + x^2} \\ \qquad x^2 \\ \quad \underline{-x^2 + x} \\ \qquad\quad x - 1 \\ \qquad \underline{-x + 1} \\ \qquad\qquad 0 \end{array}$$

3. Számítsa ki az alábbi határozatlan integrálokat!

a) $\int \frac{x^2}{x^2-9}\,\mathrm{dx}$

b) $\int \frac{1}{(x-1)^2}\,\mathrm{dx}$

c) $\int \frac{1}{x^2-1}\,\mathrm{dx}$

d) $\int \frac{x^2}{x^3-1}\,\mathrm{dx}$

e) $\int \frac{1}{x^3-1}\,\mathrm{dx}$

Megoldás:

a) $\int \frac{x^2}{x^2-9}\,\mathrm{dx} = \int \frac{x^2-9+9}{x^2-9}\,\mathrm{dx} = \int 1-9\frac{1}{(x-3)(x+3)}\,\mathrm{dx} = x-9\int \frac{1}{(x-3)(x+3)}\,\mathrm{dx}$
Ekkor

$$\frac{1}{(x-3)(x+3)} = \frac{A}{x-3} + \frac{B}{x+3}$$
$$1 = A(x+3) + B(x-3)$$

Ha $x=3$, $1 = A6$, $\Rightarrow A = \frac{1}{6}$, *valamint ha* $x=-3$, $1 = B(-6)$, $\Rightarrow B = -\frac{1}{6}$. *Ebből:*

$$\int \frac{1}{(x-3)(x+3)}\,\mathrm{dx} = \int \frac{\frac{1}{6}}{(x-3)} - \frac{\frac{1}{6}}{(x+3)}\,\mathrm{dx} =$$
$$= \frac{1}{6}\int \frac{1}{(x-3)} - \frac{1}{(x+3)}\,\mathrm{dx} = \frac{1}{6}(\ln|x-3| - \ln|x+3|) + C$$

Így:

$\int \frac{x^2}{x^2-9}\,\mathrm{dx} = x - \frac{9}{6}\ln|x-3| + \frac{9}{6}\ln|x+3| + C = x - \ln\left|\frac{x+3}{x-3}\right|^{\frac{3}{2}} + C$

b) $\int \frac{1}{(x-1)^2}\,\mathrm{dx} = \int (x-1)^{-2}\,\mathrm{dx} = \frac{(x-1)^{-1}}{-1} + C = \frac{1}{1-x} + C$

c) $\int \frac{1}{x^2-1}\,\mathrm{dx} = \int \frac{1}{(x-1)(x+1)}\,\mathrm{dx}$ *Ekkor*

$$\frac{1}{(x-1)(x+1)} = \frac{A}{x-1} + \frac{B}{x+1}$$
$$1 = A(x+1) + B(x-1)$$

Ha $x=1$, $1 = A2$, $\Rightarrow A = \frac{1}{2}$, *valamint ha* $x=-1$, $1 = B(-2)$, $\Rightarrow B = -\frac{1}{2}$. *Ebből:*

$\int \frac{1}{(x-1)(x+1)}\,\mathrm{dx} = \int \frac{\frac{1}{2}}{x-1} - \frac{\frac{1}{2}}{x+1}\,\mathrm{dx} = \frac{1}{2}(\ln|x-1| - \ln|x+1|) + C =$
$= \ln\sqrt{\left|\frac{x-1}{x+1}\right|} + C$

d) $\int \frac{x^2}{x^3-1}\,\mathrm{dx} = \int \frac{1}{3}\overbrace{\frac{3x^2}{x^3-1}}^{\int \frac{f'}{f} = \ln|f|}\,\mathrm{dx} = \frac{1}{3}\ln|x^3-1| + C =$
$= \ln\sqrt[3]{|x^3-1|} + C$

e) $\int \frac{1}{x^3-1}\,\mathrm{dx} = \int \frac{1}{(x-1)(x^2+x+1)}\,\mathrm{dx}$ *Ekkor*

$$\frac{1}{(x-1)(x^2+x+1)} = \frac{A}{x-1} + \frac{Bx+C}{x^2+x+1}$$
$$1 = A(x^2+x+1) + Bx(x-1) + C(x-1)$$

Ha $x=1$, $1=A3$, $\Rightarrow A=\frac{1}{3}$, *ha* $x=0$,
$1=A+C(-1)=\frac{1}{3}-C$, $\Rightarrow C=-\frac{2}{3}$. *Valamint:*

$$\begin{aligned}
1 =& \frac{1}{3}(x^2+x+1)+B(x^2-x)-\frac{2}{3}(x-1) \\
1 =& \frac{1}{3}x^2+\frac{1}{3}x+\frac{1}{3}+Bx^2-Bx-\frac{2}{3}x+\frac{2}{3} \\
1 =& \left(\frac{1}{3}+B\right)x^2+\left(\frac{1}{3}-B-\frac{2}{3}\right)x+1
\end{aligned}$$

Ahonnan $B=-\frac{1}{3}$

$$\int \frac{1}{(x-1)(x^2+x+1)}\,\mathrm{dx} = \int \frac{\frac{1}{3}}{x-1}+\frac{-\frac{1}{3}x-\frac{2}{3}}{x^2+x+1}\,\mathrm{dx} =$$
$$= \frac{1}{3}\int \frac{1}{x-1}-\frac{x+2}{x^2+x+1}\,\mathrm{dx} = \frac{1}{3}\ln|x-1|-\frac{1}{3}\int \frac{1}{2}\frac{2x+1}{x^2+x+1}+\frac{3}{2}\frac{1}{x^2+x+1}\,\mathrm{dx} =$$
$$= \frac{1}{3}\ln|x-1|-\frac{1}{6}\ln|x^2+x+1|-\frac{1}{2}\int \frac{1}{\left(x+\frac{1}{2}\right)^2+\frac{3}{4}}\,\mathrm{dx}$$

Ahol az utolsó tag:

$$\int \frac{1}{\left(x+\frac{1}{2}\right)^2+\frac{3}{4}}\,\mathrm{dx} = \int \frac{4}{3}\frac{1}{\frac{\left(x+\frac{1}{2}\right)^2}{\frac{3}{4}}+1}\,\mathrm{dx} = \frac{4}{3}\int \frac{1}{\left(\frac{2x+1}{\sqrt{3}}\right)^2+1}\,\mathrm{dx} =$$

$$= \frac{4}{3}\frac{\arctan\left(\frac{2x+1}{\sqrt{3}}\right)}{\frac{2}{\sqrt{3}}}+C = \frac{2}{\sqrt{3}}\arctan\left(\frac{2x+1}{\sqrt{3}}\right)+C$$

Így:

$$\int \frac{1}{x^3-1}\,\mathrm{dx} = \frac{1}{3}\ln|x-1|-\frac{1}{6}\ln|x^2+x+1|-\frac{1}{\sqrt{3}}\arctan\left(\frac{2x+1}{\sqrt{3}}\right)+C$$